

A CONSTRUCTION OF TWO DISTINCT CANONICAL SETS OF LIFTS OF BRAUER CHARACTERS OF A p -SOLVABLE GROUP

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ABSTRACT. In [5], Navarro defines the set $\text{Irr}(G \mid Q, \delta) \subseteq \text{Irr}(G)$, where Q is a p -subgroup of a p -solvable group G , and shows that if δ is the trivial character of Q , then $\text{Irr}(G \mid Q, \delta)$ provides a set of canonical lifts of $\text{IBr}_p(G)$, the irreducible Brauer characters with vertex Q . Previously, in [2], Isaacs defined a canonical set of lifts $B_\pi(G)$ of $I_\pi(G)$. Both of these results extend the Fong-Swan Theorem to π -separable groups, and both construct canonical sets of lifts of the generalized Brauer characters. It is known that in the case that $2 \in \pi$, or if $|G|$ is odd, we have $B_\pi(G) = \text{Irr}(G \mid Q, 1_Q)$. In this note we give a counterexample to show that this is not the case when $2 \notin \pi$. It is known that if $N \triangleleft G$ and $\chi \in B_\pi(G)$, then the constituents of χ_N are in $B_\pi(N)$. However, we use the same counterexample to show that if $N \triangleleft G$, and $\chi \in \text{Irr}(G \mid Q, 1_Q)$ is such that $\theta \in \text{Irr}(N)$ and $[\theta, \chi_N] \neq 0$, then it is not necessarily the case that $\theta \in \text{Irr}(N)$ inherits this property.

1. Introduction. Let p be a prime, G a finite p -solvable group, and for a class function α of G , let α^0 denote the restriction of α to the p -regular elements of G . The celebrated Fong-Swan Theorem asserts that if φ is an irreducible Brauer character of G for the prime p , then there necessarily exists an ordinary irreducible character χ of G such that $\chi^0 = \varphi$. Such a character χ is called a lift of φ . In [2], Isaacs constructs a canonical set of lifts $B_{p'}(G) \subseteq \text{Irr}(G)$, such that for each Brauer character φ of G , there is a unique lift of φ in $B_{p'}(G)$. Similarly, in [5], Navarro constructs another canonical set $\text{Irr}(G \mid Q, 1_Q)$ (which we will denote by $N_{p'}(G)$) of lifts of the irreducible Brauer characters of G . Navarro conjectured but did not prove that $B_{p'}(G) \neq N_{p'}(G)$. In this paper we give conditions under which $B_{p'}(G) = N_{p'}(G)$, and give an example to show that these sets need not be equal in general.

Let π be a set of primes, and denote by π' the complement of π . (In the classical case, π is the complement of the prime p .) Let G be a π -separable group. In [1], Gajendragadkar constructs a certain class of characters, called the π -special characters of G . An irreducible character χ is π -special if (a) $\chi(1)$ is a π -number and (b) if whenever $S \triangleleft \triangleleft G$ and $\theta \in \text{Irr}(S)$ lies under χ , then the order of $\det(\theta)$, as a character of S/S' , is a π -number. These π -special characters are known to have many interesting properties, and they are necessary for constructing the sets of characters under discussion in this paper. In particular, if $\alpha \in \text{Irr}(G)$ is π -special and $\beta \in \text{Irr}(G)$ is π' -special, then $\alpha\beta \in \text{Irr}(G)$ and this factorization is unique. If $\chi \in \text{Irr}(G)$ can be written in this factored form, we say that χ is π -factorable and we let χ_π and $\chi_{\pi'}$ be the π -special and π' -special factors of χ , respectively. In addition, if $M \triangleleft G$ has π' -index in G , and if $\gamma \in \text{Irr}(M)$ is π -special and invariant in G , then there is a unique π -special character $\alpha \in \text{Irr}(G)$ such that α extends γ . If $\gamma \in \text{Irr}(M)$ is π -special and $M \triangleleft G$ with G/M is a π -group, then every character $\alpha \in \text{Irr}(G \mid \gamma)$ is π -special.

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We now briefly review Isaacs' construction of $B_\pi(G)$ in [2] and Navarro's construction of the similar set, $\text{Irr}(G|Q, 1_Q)$. In [2] Isaacs proves that if G is π -separable, and if $\chi \in \text{Irr}(G)$, then there is a unique (up to conjugacy) pair (S, φ) maximal with the property that $S \triangleleft \triangleleft G$, $\varphi \in \text{Irr}(S)$ lies under χ , and φ is π -factorable. Such a pair (S, φ) is called a maximal factorable subnormal pair. Denote by T the stabilizer of φ in the normalizer in G of S . Isaacs then shows that if $S \neq G$, then $T < G$ and induction defines a bijection from $\text{Irr}(T | \varphi)$ to $\text{Irr}(G | \varphi)$. Therefore, we can let $\psi \in \text{Irr}(T)$ be the unique character of T lying over φ such that $\psi^G = \chi$. The subnormal nucleus (W, γ) of χ is then defined recursively by defining (W, γ) to be the subnormal nucleus of ψ . (If χ is π -factorable, then (W, γ) is defined to be (G, χ) .) Note that by the construction, $\gamma^G = \chi$ and γ is π -factorable. Also, it is shown that the subnormal nucleus of χ is unique up to conjugacy. The set $B_\pi(G)$ is defined as the set of irreducible characters of G whose nucleus character γ is π -special.

In [5], Navarro similarly defines the set $\text{Irr}(G|Q, 1_Q)$ for a p -solvable group G . Navarro shows that if $\chi \in \text{Irr}(G)$, then there is a unique pair (N, θ) maximal with the property that $N \triangleleft G$, $\theta \in \text{Irr}(N)$ lies under χ , and θ is p -factorable. Such a pair is called a maximal factorable normal pair. It is shown that if $N < G$, then $G_\theta < G$, and thus the Clifford correspondence implies that if $T = G_\theta$, there is a unique character $\psi \in \text{Irr}(T | \theta)$ such that $\psi^G = \chi$. Navarro then defines the normal nucleus (U, ϵ) to be the normal nucleus of (T, ψ) . (Again, if χ is p -factorable, then $(U, \epsilon) = (G, \chi)$.) Note that again $\epsilon^G = \chi$ and ϵ is p -factorable. If Q is a Sylow p -subgroup of U , and if $\delta \in \text{Irr}(Q)$ is defined by $\delta = (\epsilon_p)_Q$, then we say the pair (Q, δ) is a normal vertex for χ , and it is shown that this pair and the normal nucleus are unique up to conjugacy. The set $\text{Irr}(G | Q, 1_Q)$ is defined as $\{\chi \in \text{Irr}(G) \mid \delta = 1_Q\}$, or equivalently, the set of irreducible characters of G with a p' -special normal nucleus character. Although Navarro only defines the set $\text{Irr}(G|Q, \delta)$ when $\pi = p'$ and G is p -solvable, the same construction of the normal nucleus and vertex of a character works if π is an arbitrary set of primes and G is π -separable. In this case, we will define the set $N_\pi(G)$ to be $\text{Irr}(G|Q, 1_Q)$, only now Q is a Hall π' -subgroup of the normal nucleus subgroup U of χ . Thus $B_\pi(G)$ consists of those irreducible characters of G with a π -special subnormal nucleus character, and $N_\pi(G)$ consists of those characters of G with a π -special normal nucleus character.

Recall that if χ is any class function of the π -separable group G , then χ^0 denotes the restriction of χ to the elements of G whose order is a π -number. Moreover, the set $I_\pi(G)$ is a generalization of Brauer characters in a p -solvable group G to a set of primes π (so that if $\pi = p'$, and if G is p -solvable, then the set $I_\pi(G)$ is exactly the set of Brauer characters of G for the prime p). In [2] it is shown that if $\chi \in B_\pi(G)$, then $\chi^0 \in I_\pi(G)$, and in [5] it is shown that if $\eta \in N_\pi(G)$, then $\eta^0 \in I_\pi(G)$. Both $B_\pi(G)$ and $N_\pi(G)$ are canonical sets of lifts of $I_\pi(G)$; in other words, if $\varphi \in I_\pi(G)$, then there is a unique character $\chi \in B_\pi(G)$ and a unique character $\eta \in N_\pi(G)$ such that $\chi^0 = \eta^0 = \varphi$. Moreover, Isaacs shows that if $\chi \in B_\pi(G)$ and $N \triangleleft G$, then every constituent θ of χ_N is in $B_\pi(N)$.

We will need the following results. In [4], Isaacs defines, for a subgroup H of a π -separable group G (where $2 \notin \pi$), a linear character $\delta_{(G:H)} \in \text{Irr}(H)$, called the standard sign character of H . This theorem lists some of the properties of $\delta_{(G:H)}$.

Theorem 1.1. *Let G be π -separable with $2 \notin \pi$, and let δ be the standard sign character for some subgroup H of G . Then the following hold:*

- (1) If $|G : H|$ is a π -number and H is a maximal subgroup of G , then δ is the permutation sign character of the action of H on the right cosets of H in G .
- (2) $\text{core}_G(H) \subseteq \ker(\delta)$.
- (3) Suppose $H \subseteq G$ has π index. If $\psi \in \text{Irr}(H)$ and $\psi^G = \chi \in \text{Irr}(G)$, then χ is π -special if and only if $\delta\psi$ is π -special.

Proof. This is the content of Theorems 2.5 and B of [4] and Lemma 2.1 of [3]. \square

The aims of this paper, then, are threefold. First, we prove the statement made (without proof) in [5] that if p is an odd prime, the sets $B_{p'}(G)$ and $N_{p'}(G)$ coincide. Secondly, we give a counterexample to show that if $2 \in \pi'$, then $B_\pi(G)$ and $N_\pi(G)$ need not coincide. Finally, we use the same counterexample to show that if $\eta \in N_\pi(G)$ and $M \triangleleft G$, then it need not be the case that the constituents of η_M are themselves in $N_\pi(M)$.

2. Equality occurs if G has odd order. In this section we give a brief proof that if $|G|$ is odd or if $2 \in \pi$, then $B_\pi(G) = N_\pi(G)$. This result was stated without proof in [5]. Here the field \mathbb{Q}_π is defined by adjoining the n th roots of unity to \mathbb{Q} for every π -number n .

Lemma 2.1. *Let G be π -separable and assume $2 \in \pi$ or $|G|$ is odd. Assume $\chi \in \text{Irr}(G)$ has values in \mathbb{Q}_π and that $\chi^0 \in I_\pi(G)$. Then $\chi \in B_\pi(G)$.*

Proof. This is Theorem 12.3 of [2]. \square

Lemma 2.2. *Let G be π -separable, and suppose $\eta \in N_\pi(G)$. Then $\eta(g) \in \mathbb{Q}_\pi$ for all elements $g \in G$.*

Proof. Let σ be any automorphism of \mathbb{C} that fixes the n th roots of unity for every π -number n . Clearly $\eta^\sigma \in N_\pi(G)$ by the construction of $N_\pi(G)$. If $\eta^0 = \varphi \in I_\pi(G)$, then since the values of φ are in \mathbb{Q}_π , then $\varphi^\sigma = \varphi$. Since η is the unique lift of φ in $N_\pi(G)$, then it must be that $\eta^\sigma = \eta$. Thus the values of η are in \mathbb{Q}_π . \square

Corollary 2.3. *If G is π -separable and $2 \in \pi$ or $|G|$ is odd, then $B_\pi(G) = N_\pi(G)$.*

Proof. Since $B_\pi(G)$ and $N_\pi(G)$ are both sets of lifts of $I_\pi(G)$, then $|B_\pi(G)| = |N_\pi(G)|$. By Lemmas 2.1 and 2.2, $N_\pi(G) \subseteq B_\pi(G)$. Thus $N_\pi(G) = B_\pi(G)$. \square

3. A counterexample of even order. In this section we construct the aforementioned counterexample to show that $B_\pi(G)$ need not equal $N_\pi(G)$, and we show that the constituents of the restriction of a character in $N_\pi(G)$ to a normal subgroup V need not be in $N_\pi(V)$.

Suppose Γ is a finite group of order n , and let p be a prime number. Let E be an elementary abelian group of order p^n . Then we can let Γ act on E by associating to each element $x \in \Gamma$ one of the cyclic factors of E , and let the left multiplication action of Γ on itself induce an action on the factors of E . Let G be the semidirect product of Γ acting on E with this action. Then for each subgroup L of G such that $E \subseteq L \subseteq G$, we see that there is an irreducible character θ of E such that $G_\theta = L$.

Construction: Let S_1 be isomorphic to the symmetric group on four elements, and let A_1 be isomorphic to the alternating group on four elements. Define the group U_1 as the semidirect product of S_1 acting on A_1 with the conjugation action, so that $U_1 = A_1 S_1$ and $S_1 \cap A_1 = 1$. Let K_1 be the normal Klein four group in A_1 , and note that $K_1 \triangleleft U_1$. Define the element $x \in S_1$ by setting x equal to the permutation (12). Note that $\langle A_1, x \rangle \cong \text{Sym}(4)$.

Let $H_1 = \langle A_1, x \rangle$, and notice that H_1 is not subnormal in U_1 , $K_1 \triangleleft H_1$, and $H_1/K_1 \cong \text{Sym}(3)$. Let $L_1 \subseteq U_1$ be such that $K_1 \subseteq L_1 \subseteq H_1$ and $|H_1 : L_1| = 3$. Note that $L_1 \cap A_1 = K_1$ and $L_1 A_1 = H_1$. Moreover, note that $\mathbb{O}_3(U_1) = 1$.

Let U_2 be isomorphic to U_1 , set $V_0 = U_1 \times U_2$, and define Γ as the semidirect product of \mathbf{C}_2 acting on V_0 , with the nontrivial element of \mathbf{C}_2 acting by interchanging the components of V_0 . Note that $\text{core}_{\Gamma}(L_1) = 1$, and $\mathbb{O}_3(\Gamma) = 1$.

Let $|\Gamma| = n$, and let E be an elementary abelian group of order 3^n . Let G be the semidirect product as defined in the paragraph preceding this construction, so that $G/E \cong \Gamma$. Therefore there is a character $\theta \in \text{Irr}(E)$ such that $G_{\theta} = EL_1$. Set $L = EL_1$, $A = EA_1$, $K = EK_1$, $H = EH_1$, and $V = EV_0$.

Set $\pi = \{3\}$ and note that G is solvable. Since E is a 3-group, then θ is π -special. Note that the π -special character θ extends to a unique π -special character $\hat{\theta} \in \text{Irr}(K)$. We see that since θ and $\hat{\theta}$ uniquely determine each other, then $A_{\hat{\theta}} = K$. Therefore $\hat{\theta} \in \text{Irr}(K)$ induces irreducibly to a π -special character $\varphi \in \text{Irr}(A)$. Since φ is uniquely determined by θ , and L normalizes A , then $L = G_{\theta} \subseteq G_{\varphi}$.

We now show that $G_{\varphi} = H$. By a Frattini argument, we see that $G_{\varphi} \subseteq G_{\theta}A = H$. Since $\varphi \in \text{Irr}(A)$, then clearly $A \subseteq G_{\varphi}$. We showed in the previous paragraph that $L = G_{\theta} \subseteq G_{\varphi}$. Thus $H = AL \subseteq G_{\varphi}$ and therefore $G_{\varphi} = H$.

We now claim that (E, θ) is a maximal factorable normal pair. Recall that $\mathbb{O}_3(\Gamma)$ is trivial, and thus $\mathbb{O}_3(G/E)$ is trivial. Suppose there exists a factorable normal pair (N, ψ) such that $E \subseteq N$ and θ lies under ψ , and suppose N/E is a nontrivial 2-group. Then the π -special factor of ψ must restrict to θ , and thus θ must be invariant in N . Therefore $N \subseteq G_{\theta} = L$. However, $\text{core}_{G/E}(L/E)$ is trivial, and this yields a contradiction. Thus (E, θ) is a maximal factorable normal pair.

We also claim that (A, φ) is a maximal factorable subnormal pair. Suppose (S, σ) is a factorable subnormal pair such that $A \triangleleft S$ and σ lies over φ , and suppose S/A is a nontrivial 2-group. Since $\varphi \in \text{Irr}(A)$ is π -special, φ must be invariant in S , and thus $S \subseteq G_{\varphi} = H$. Since $|H : A| = 2$ and H is not subnormal in G , we have a contradiction. Since there are no subnormal subgroups T of G such that $A \triangleleft T$ and T/A is a nontrivial 3-group, then (A, φ) is a maximal factorable subnormal pair.

Note that since $|L : K| = 2$ and the π -special character $\hat{\theta} \in \text{Irr}(K)$ is invariant in L , then $\hat{\theta}$ extends to a π -special character $\xi \in \text{Irr}(L)$, and we define $\eta \in \text{Irr}(G)$ by $\eta = \xi^G$. Note that θ lies under η , and since (E, θ) is a maximal factorable normal pair, and the Clifford correspondent $\xi \in \text{Irr}(G_{\theta}|\theta)$ for η is π -special, then $\eta \in \text{N}_{\pi}(G)$.

We now show that η is not in $\text{B}_{\pi}(G)$. Note that $\xi^H \in \text{Irr}(H)$, and since $\varphi \in \text{Irr}(A)$ is exactly $(\xi_K)^A = (\xi^H)_A$, then φ lies under both ξ^H and η . Recall that (A, φ) is a maximal factorable subnormal pair, and $H = G_{\varphi}$. Since $|H : A| = 2$ and $\varphi \in \text{Irr}(A)$ is π -special, every character of H lying over φ must be π -factorable. Thus (H, ξ^H) is a subnormal nucleus for η , and $\eta \in \text{B}_{\pi}(G)$ if and only if ξ^H is π -special.

Let $\delta = \delta_{(H:L)} \in \text{Irr}(L)$ be the standard sign character for $L \subseteq H$. Note that $K \subseteq \ker(\delta)$. Therefore we see that δ is the nonprincipal linear character of L/K . Since $\xi \in \text{Irr}(L)$ is π -special and $\delta \in \text{Irr}(L)$ is π' -special, then $\xi\delta \in \text{Irr}(L)$ is not π -special. Thus ξ^H is not π -special by Theorem 1.1. Therefore, since (H, ξ^H) is a subnormal nucleus for η , then η is not in $\text{B}_{\pi}(G)$.

Finally, note that (H, ξ^H) is a normal nucleus for ξ^V , which lies under η . Since ξ^H is not π -special, then ξ^V is not π -special, and the constituents of η_V are not in $N_\pi(V)$. \square

The above example raises some interesting questions. Are there other families of characters which form lifts of the set $I_\pi(G)$? Some results in this direction can be found in forthcoming papers of Mark Lewis. What in general can be said about the set of lifts of a Brauer character of a p -solvable group G ? In a future paper, the author will describe some bounds on the number of lifts of a Brauer character of a p -solvable group. There is still much to be known, though, about the set of lifts of a Brauer character of a p -solvable group.

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REFERENCES

- [1] D. Gajendragadkar, A characteristic class of characters of finite π -separable groups, *J. Algebra* **59** (1979), 237-259.
- [2] I.M. Isaacs, Characters of π -separable groups, *J. Algebra* **86** (1984), 98-128.
- [3] I. M. Isaacs, Induction and restriction of π -partial characters and their lifts, *Canad. J. Math.* **48** (1996), 1210-1223.
- [4] I. M. Isaacs, Induction and restriction of π -special characters, *Canad. J. Math.* **37** (1986), 576-604.
- [5] G. Navarro, Vertices for characters of p -solvable groups, *Trans. Amer. Math. Soc.* **354** (2002), 2759-2773.

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